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INVERSE LIMITS OF RINGS AND MULTIPLIER RINGS

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Abstract. It is proved that the exchange property, the Bass stable rank and the quasi-Bass property are all preserved under surjective inverse limits. This is then applied to multiplier rings by showing that these in many cases can be obtained as inverse limits.

Introduction

Given a sequence (R_n) of rings with connecting morphisms (i.e. ring homomorphisms) $\pi_n\colon R_n \to R_{n-1}$ for all n (setting $R_0 = R_1$ and $\pi_1 = \mathrm{id}$) we define the *inverse limit* as the ring $\varprojlim R_n$ of strings $x = (x_n)$ in $\prod R_n$, i.e. sequences such that $\pi_n(x_n) = x_{n-1}$ for all n. For each m there is a natural morphism $\rho_m\colon \varprojlim R_n \to R_m$ (the *coordinate evaluation*) obtained by evaluating a string $x = (x_n)$ at m, and we see that $\pi_n \circ \rho_n = \rho_{n-1}$ for every n. The ring $R = \varprojlim R_n$ has the universal property that for each coherent sequence of morphisms $\sigma_n\colon S \to R_n$ from a ring S (i.e. $\pi_n \circ \sigma_n = \sigma_{n-1}$ for all n) there is a unique morphism $\sigma\colon S \to R$ such that $\sigma_n = \rho_n \circ \sigma$ for all n.

If for each m we let $S_m = \rho_m(\varprojlim R_n) \subset R_m$, then $\pi_n(S_n) = S_{n-1}$ and $\varprojlim R_n = \varprojlim S_n$. This shows that every ring which can be obtained as an inverse limit can also be obtained as an inverse limit in which each morphism π_n is surjective. We shall refer to this case as a *surjective inverse limit*, and will concentrate exclusively on it.

By their very construction inverse limits tend to be large. For example, if the morphisms are not eventually constant, an inverse limit will be uncountable even if the rings are finite. This might be considered a detracting factor. In this paper we shall try to redeem the construction by showing that in many instances multiplier rings (which we expect to be large) can be obtained as inverse limits. Since the structural properties of inverse limits are good, we hereby obtain information about multiplier rings that would otherwise seem unreachable.

We prove in Section 1 that a surjective inverse limit of exchange rings is again an exchange ring. In Section 2 we show that the Bass stable rank of a surjective inverse limit of rings (R_n) is the supremum of the Bass stable ranks of the R_n's. In Section 3 we extend this to an important infinite case by proving that the surjective inverse limit of quasi-Bass rings is again a quasi-Bass ring.

In Sections 4 and 5 we establish the basic properties of approximate units and multiplier rings that we shall need. In particular we show that for every proper morphism $\varphi\colon R \to S$ between non-degenerate rings there is a unique extension $\overline{\varphi}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ between their multiplier rings. We also show that $\overline{\varphi}$ is strictly continuous, cf. [9]. If R is $\sigma-$unital, i.e.

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has a countable approximate unit, and φ is surjective we show that $\overline{\varphi}$ is surjective as well, thereby giving an algebraic analogue of the Tietze extension theorem (see e.g. [18]).

Finally in Section 6 we combine our results by showing that many multiplier rings can be obtained as inverse limits. Thus if (R_n) is a sequence of $\sigma-$unital rings with surjective morphisms $\pi_n\colon R_n \to R_{n-1}$ we prove that $\mathcal{M}(\varprojlim R_n) = \varprojlim \mathcal{M}(R_n)$. Going further we show that if a semi-prime $\sigma-$unital ring R has two sequences of ideals (I_n) and (J_n), one increasing to R, the other decreasing to 0, such that $I_n \cap J_n = 0$ and R/J_n is unital for every n, then $\mathcal{M}(R) = \varprojlim R/J_n$.

The inspiration for this paper comes from [15], where a similar list of results are obtained for C^*-algebras. In this category one must of course consider only *bounded strings* as elements in the inverse limit, so although our results are the same as in [15], our proofs are completely different - sometimes harder, sometimes easier. It is worth mentioning that inverse limits of C^*-algebras in the algebraic sense (with arbitrary, unbounded strings) have been explored by Phillips in [34] and [35] to obtain examples of *pro* C^*-*algebras*. The aim was to create a non-commutative analogue of normal spaces that are not necessarily locally compact. The unpublished notes [20] by Goodearl have also influenced our work, although we fail to answer the question that motivated them: Will a surjective inverse limit of separative exchange rings itself be separative? A negative answer would provide a solution to the Fundamental Separativity Problem (see [4]).

1. Exchange Rings

Recall from [21] and [28] that a (unital) ring R is an *exchange ring* if for each x in R there is an idempotent e in xR such that $1 - x = (1 - e)(1 - y)$ for some y in R. This is not the original definition, which concerns a finite exchange property for $R-$modules, see [39], but an equivalent formulation better suited for our purposes.

The class of exchange rings is pleasantly large and includes all (von Neumann) regular rings, all $\pi-$regular rings, the semi-perfect rings (identified with the semi-local exchange rings) and the C^*-algebras of real rank zero (identified with the exchange C^*-algebras by [4, Theorem 7.2]).

We shall show that every surjective inverse limit of exchange rings again has the exchange property. As an hors d'œuvre we present the following simpler result from [20] (see also [19, Example 1.10]). Recall that an element x in a ring R is said to be *regular* provided that $x = xyx$ for some y in R. We shall refer to such an element y as a *partial inverse* for x. If all elements of R are regular we say that R is *(von Neumann) regular*.

Proposition 1.1. (Goodearl) *If R is the surjective inverse limit of a sequence (R_n) of regular rings, then R is also regular.*

Proof. Put $R = \varprojlim R_n$ and let $\pi_n\colon R_n \to R_{n-1}$ denote the connecting morphisms. If $x = (x_n)$ is an element in R and if for all $k < n$ we have found elements y_k in R_k such that $\pi_k(y_k) = y_{k-1}$ and $x_k y_k x_k = x_k$, we choose an element y in R_n with $\pi_n(y) = y_{n-1}$. Then $u = x_n - x_n y x_n \in \ker \pi_n$, and since this ideal is a regular ring in its own right we can find a

v in $\ker \pi_n$ such that $uvu = u$. By computation

$$\begin{aligned} x_n - x_n y x_n &= (x_n - x_n y x_n) v (x_n - x_n y x_n) \\ &= x_n v x_n - x_n v x_n y x_n - x_n y x_n v x_n + x_n y x_n v x_n y x_n \,, \end{aligned}$$

and it follows that the element $y_n = y + v - yx_n v - vx_n y + yx_n vx_n y_n$ is a partial inverse for x_n with $\pi_n(y_n) = y_{n-1}$.

By induction we can therefore find an element $y = (y_n)$ in R such that $x = xyx$, so every element in R is regular, as desired. □

Note that in the definition of an exchange ring the role of the unit is superfluous. Thus we say that a (not necessarily unital) ring R is an exchange ring if for each x in R there is an idempotent e in xR such that $x = e + y - ey$ for some y in R. This idea was successfully exploited by Ara in [1]. If I is a non-unital exchange ring embedded as a two-sided ideal of a unital ring R, we will adopt the terminology from [1] and say that I is an *exchange ideal of* R.

Lemma 1.2. *Let I be a two-sided exchange ideal in a unital ring R. If x and y are elements in R and p is an idempotent such that $p - xy \in I$, there is an idempotent q in I and an element r in pRp with $p - r$ in I, such that both elements $a = (p - q)x$ and $b = yr$ are regular and partial inverses for one another. By construction $a - px \in I$ and $b - yp \in I$, so $ab - p \in I$.*

This is a restatement of [1, Lemma 2.1]. The result is obtained by applying the exchange condition to the element $p - pxyp$ in pIp to get q.

If R is a non-unital ring, it is sometimes convenient to adjoin a unit in order to obtain the ring $R^+ = R \oplus \mathbb{Z}$, with elementwise addition and multiplication given by the rule $(x, n)(y, m) = (xy + mx + ny, nm)$. In this way R sits as a two-sided ideal in R^+ with quotient $\mathbb{Z}$.

Lemma 1.3. *Let $\pi \colon R \to S$ be a surjective morphism between (not necessarily unital) exchange rings, and let $\overline{x}, \overline{y}$ and $\overline{z}$ be elements in S such that with $\overline{e} = \overline{x}\,\overline{y}$ we have*

$$\overline{y} = \overline{y}\,\overline{e}, \qquad 1 - \overline{e} = (1 - \overline{x})(1 - \overline{z}), \qquad (1 - \overline{z})\overline{e} = 0 \,.$$

For each choice of x in R with $\pi(x) = \overline{x}$ there are then elements y and z in R with $\pi(y) = \overline{y}$ and $\pi(z) = \overline{z}$, such that with $e = xy$ we have

$$y = ye, \qquad 1 - e = (1 - x)(1 - z), \qquad (1 - z)e = 0 \,.$$

Proof. This proof amalgamates arguments from [1, Theorem 2.2] and [28, Proposition 1.1], which are included for convenience. As usual we shall write $p \leq q$ for idempotents p and q such that $pq = qp = p$. Moreover, if R is not unital we adjoin a unit to obtain the unital ring $R^+ = R \oplus \mathbb{Z}$ and we identify S with an exchange ideal in S^+, setting $\pi(1) = 1$.

Observe first that $\overline{e}$ is an idempotent by necessity, since $\overline{e}\,\overline{e} = \overline{x}\,\overline{y}\,\overline{e} = \overline{x}\,\overline{y} = \overline{e}$. Thus also $\overline{p}_1 = \overline{y}\,\overline{x}$ is an idempotent. Since R is an exchange ring $\overline{p}_1$ can be lifted to an idempotent p_1 in R by [1, Theorem 2.2]. Let y_1 and z_1 be any lifts in R of the elements $\overline{y}$ and $\overline{z}$. By Lemma 1.2 (with $I = \ker(\pi)$ and exchanging the roles of x and y) we can find regular elements a_1 and b_1 in R that are partial inverses for one another, such that $a_1 = (p_1 - q_1)y_1$ and $b_1 = xr_1$

for some idempotent q_1 in I and some r_1 in p_1Rp_1 with $\pi(r_1) = \overline{p}_1$. Thus $e_1 = b_1a_1$ is an idempotent with

$$\pi(e_1) = \overline{x}\,\overline{p}_1\overline{p}_1\overline{y} = \overline{e}\,\overline{e}\,\overline{e} = \overline{e}\,.$$

Put $z_2 = 1 - (1 - z_1)(1 - e_1)$, so that $\pi(z_2) = 1 - (1 - \overline{z})(1 - \overline{e}) = \overline{z}$. Observe that $\overline{p}_2 = (1 - \overline{z})(1 - \overline{x})$ is an idempotent in $1 - S$ and lift it to an idempotent p_2 in $1 - R$. Again by Lemma 1.2 there are regular elements a_2 and b_2 in $1 - R$, partial inverses for one another, such that $a_2 = (p_2 - q_2)(1 - z_2)$ and $b_2 = (1 - x)r_2$ for some idempotent q_2 in I and some element r_2 in $p_2R^+p_2$ with $\pi(r_2) = \overline{p}_2 = (1 - \overline{z})(1 - \overline{x})$ (so that $r_2 \in 1 - R$ as well).

The idempotent $f = b_2a_2$ belongs to $(1 - x)(1 - R)(1 - z_2)$ and satisfies

$$\pi(f) = (1 - \overline{x})\overline{p}_2(1 - \overline{z}) = (1 - \overline{x})(1 - \overline{z})(1 - \overline{x})(1 - \overline{z}) = 1 - \overline{e}\,.$$

Since $f = f(1 - e_1)$ by construction of z_2, we may consider the idempotent $(1 - e_1)f$, where now $(1 - e_1)f \le 1 - e_1$. Thus $q = (1 - e_1)(1 - f)$ is an idempotent in I with $q \le 1 - e_1$.

Now apply the exchange property to the element qxq in the corner ring $qRq = qR^+q\,(\subset I)$ to find an idempotent t in qRq such that $t = qxqcq$ and $q - t = q(1 - x)qdq$ for some elements c, d in R. Put $s = xqct$, and note that $sq = s$ and $qs = t$, whence $s^2 = s$. Since $t \le q \le 1 - e_1$, the element $e_2 = e_1 + (1 - e_1)s$ is an idempotent in R with $e_1 \le e_2$ and $\pi(e_2) = \overline{e}$. Let $y_2 = r_1a_1(1 - s) + qct$. Then $\pi(y_2) = \overline{p}_1\overline{y} = \overline{y}\,\overline{x}\,\overline{y} = \overline{y}$. Moreover,

$$xy_2 = b_1a_1(1 - s) + s = e_1(1 - s) + s = e_1 + (1 - e_1)s = e_2\,. \tag{1}$$

The argument in [1, Theorem 2.2] proceeds to show that the right ideal $A = e_2R^+ + (1 - x)R^+$ equals R^+, and does so by showing that both $1 - q$ and q belong to A. All we need is the last assertion, but there seems no easy way to obtain the specific decomposition of q except by the full argument.

Evidently $e_2e_1 \in A$ and $e_2(1 - e_1) \in A$. We know that $e_2e_1 = e_1$, and therefore $e_2(1 - e_1) = e_2 - e_1 = (1 - e_1)s = s - e_1s$. Consequently both $e_1 \in A$ and $s \in A$. Now $f = (1 - x)r_2a_2 \in A$, so also $(1 - e_1)f = f - e_1f \in A$. It follows that $1 - q = e_1 + (1 - e_1)f \in A$. As $s \in A$ and $t = qs$ we now conclude that $t = s - (1 - q)s \in A$. Since

$$q - t = q(1 - x)qdq = (1 - x)qdq - (1 - q)(1 - x)qdq \in (1 - x)R + A = A\,,$$

we finally see that $q = q - t + t \in A$. We can therefore write

$$q = e_2u + (1 - x)v \tag{2}$$

for some elements u and v in Rq. In particular, both elements belong to I.

Put $z_3 = 1 - (r_2a_2 + v)$ and $e_3 = e_1(1 - f) + u$, and note that $\pi(1 - z_3) = \overline{p}_2\overline{p}_2(1 - \overline{z}) = (1 - \overline{z})(1 - \overline{x})(1 - \overline{z}) = 1 - \overline{z}$ and $\pi(e_3) = \overline{e}\,\overline{e} = \overline{e}$. Moreover, by (2)

$$\begin{aligned}(1 - x)(1 - z_3) + e_2e_3 &= b_2a_2 + (1 - x)v + e_2e_1(1 - f) + e_2u \\ &= f + e_1(1 - f) + q = f + e_1(1 - f) + (1 - e_1)(1 - f) = 1\,.\end{aligned} \tag{3}$$

We can now make our final choices as follows:

$$e = e_2 + e_2e_3(1 - e_2), \qquad y = y_2e, \qquad z = 1 - (1 - z_3)(1 - e_2)(1 - e)\,.$$

It is easy to check that

$$\pi(e) = \overline{e}, \qquad \pi(y) = \overline{y}\,\overline{e} = \overline{y}, \qquad \pi(z) = 1 - (1 - \overline{z})(1 - \overline{e})(1 - \overline{e}) = \overline{z}.$$

Moreover, by (1) and (3) we have the desired relations:

$$\begin{aligned} xy &= xy_2e = e_2e = e \\ ye &= y \quad \text{and} \quad (1-z)e = 0 \\ (1-x)(1-z) &= (1-x)(1-z_3)(1-e_2)(1-e) = (1-e_2e_3)(1-e_2)(1-e) \\ &= (1-e_2-e_2e_3(1-e_2))(1-e) = (1-e)(1-e) = 1-e\,. \end{aligned}$$

□

Theorem 1.4. *If R is the surjective inverse limit of a sequence (R_n) of exchange rings, then R is also an exchange ring.*

Proof. Consider an element x in R, identified with a string (x_n) in $\prod R_n$. We must then find an idempotent $e = (e_n)$ in R such that $e \in xR$ and $1 - e \in (1-x)(1-R)$.

Since R_1 is an exchange ring we can find an idempotent e_1 and elements y_1, z_1 such that $e_1 = x_1y_1$ and $1 - e_1 = (1-x_1)(1-z_1)$. Evidently we may also assume that $y_1 = y_1e_1$ and $(1-z_1)e_1 = 0$.

Assume now that for some n we have found elements y_k and z_k in R_k for $0 \le k \le n$, such that $\pi_k(y_k) = y_{k-1}$ and $\pi_k(z_k) = z_{k-1}$ for all $k \ge 2$, and moreover with $e_k = x_ky_k$ we have the relations

$$y_k = y_ke_k, \qquad 1-e_k = (1-x_k)(1-z_k), \qquad (1-z_k)e_k = 0,$$

for all k. By Lemma 1.3 we can then find elements y_{n+1} and z_{n+1} in R_{n+1} with $\pi_{n+1}(y_{n+1}) = y_n$ and $\pi_{n+1}(z_{n+1}) = z_n$, such that with $e_{n+1} = x_{n+1}y_{n+1}$ we have

$$y_{n+1} = y_{n+1}e_{n+1}, \qquad 1-e_{n+1} = (1-x_{n+1})(1-z_{n+1}), \qquad (1-z_{n+1})e_{n+1} = 0.$$

By induction this defines elements $y = (y_n)$ and $z = (z_n)$ in R, such that the element $e = xy$ is an idempotent and $1 - e = (1-x)(1-z)$, as desired. □

2. Bass Stable Rank

Let R be a unital ring. As usual we say that a row $\mathbf{a} = (a_1, \dots, a_d)$ in R^d is *right unimodular* provided that $a_1R + \cdots + a_dR = R$. To facilitate the computations with rows in R^d we introduce the $R-$valued inner product $\mathbf{a} \cdot \mathbf{b} = \sum a_ib_i$, so that $\mathbf{a}$ is right unimodular precisely if $\mathbf{a} \cdot \mathbf{b} = 1$ for some $\mathbf{b}$ in R^d. Also, if $\mathbf{a} = (a_1, \dots, a_d) \in R^d$ and $s \in R$, we write $s\mathbf{a} = (sa_1, \dots, sa_d)$.

Recall from [10] (see also e.g. [27], [25]) that the *Bass stable rank* of a unital ring R is the smallest number $\mathrm{bsr}(R)$ such that for each $d \ge \mathrm{bsr}(R)$ every right unimodular row $\mathbf{a} = (a_0, \dots, a_d)$ in R^{d+1} can be reduced to a right unimodular row in R^d of the form $\mathbf{a}^r + a_0\mathbf{b}$ for a suitable row $\mathbf{b}$ in R^d. Here $\mathbf{a}^r = (a_1, \dots, a_n)$ and we regard R^d as a two-sided module over R. Evidently this definition favours right unimodular rows and should be called the right Bass rank, but it turns out that the analogous concept for left unimodular rows gives the same lower bound ([37], [27]). In particular $\mathrm{bsr}(R) = 1$ (in which case we say that R is a *Bass ring*) if every equation $ax + b = 1$ implies that $a + by$ is invertible in R for a suitable y, cf. [38].

Lemma 2.1. *Let R be a unital ring with $\mathrm{bsr}(R) \le d$. Given rows $\mathbf{a}$ and $\mathbf{x}$ in R^d with $\mathbf{a}\cdot\mathbf{x} = 1 - s$ for some s in R, there exist rows $\mathbf{b}$ and $\mathbf{y}$ in R^d and z in R such that*

$$(\mathbf{a} + s\mathbf{b})\cdot(\mathbf{x} + \mathbf{y}sz) = 1\,.$$

Proof. Since $\mathrm{bsr}(R) \le d$, the equation $\mathbf{a}\cdot\mathbf{x} + s1 = 1$ produces rows $\mathbf{b}$ and $\mathbf{y}$ in R^d such that $(\mathbf{a} + s\mathbf{b})\cdot\mathbf{y} = 1$. Define the elements $y = \mathbf{a}\cdot\mathbf{y}$, $b = \mathbf{b}\cdot\mathbf{y}$ and $z = 1 - \mathbf{b}\cdot\mathbf{x}$, and note that $y + sb = 1$. By computation we therefore get

$$\begin{aligned}(\mathbf{a} + s\mathbf{b})\cdot(\mathbf{x} + \mathbf{y}sz) &= 1 - s + ysz + s(1 - z) + sbsz\\ = 1 - s + (y + sb)sz + s(1 - z) &= 1 - s + sz + s(1 - z) = 1\,,\end{aligned}$$

as desired. □

Lemma 2.2. *Consider a surjective morphism $\pi\colon R \to S$ between unital rings, where $\mathrm{bsr}(R) \le d$. Assume that $\mathbf{a}$ and $\mathbf{x}$ are unital rows in R^{d+1} such that $\mathbf{a}\cdot\mathbf{x} = 1$, and that we have chosen rows $\overline{\mathbf{b}}$ and $\overline{\mathbf{y}}$ in S^d such that $(\pi(\mathbf{a}^r) + \pi(a_0)\overline{\mathbf{b}})\cdot\overline{\mathbf{y}} = 1$. We can then find $\mathbf{b}$ and $\mathbf{y}$ in R^d such that $\pi(\mathbf{b}) = \overline{\mathbf{b}}$ and $\pi(\mathbf{y}) = \overline{\mathbf{y}}$, and moreover $(\mathbf{a}^r + a_0\mathbf{b})\cdot\mathbf{y} = 1$.*

Proof. Take any lifts $\mathbf{b}'$ of $\overline{\mathbf{b}}$ and $\mathbf{y}'$ of $\overline{\mathbf{y}}$. Then $(\mathbf{a}^r + a_0\mathbf{b}')\cdot\mathbf{y}' = 1 - t$ for some t in $I = \ker\pi$. Since $\mathrm{bsr}(R) \le d$ we can use Lemma 2.1 to find $\mathbf{s}$ and $\mathbf{t}$ in I^d such that

$$(\mathbf{a}^r + a_0\mathbf{b}' + \mathbf{s})\cdot(\mathbf{y}' + \mathbf{t}) = 1\,. \tag{4}$$

¿From the original condition, setting $b = x_0 - \mathbf{b}'\cdot\mathbf{x}^r$, we get

$$1 = \mathbf{a}^r\cdot\mathbf{x}^r + a_0x_0 = (\mathbf{a}^r + a_0\mathbf{b}')\cdot\mathbf{x}^r + a_0b\,.$$

It follows from (4) that with $s = \mathbf{s}\cdot(\mathbf{y}' + \mathbf{t})$ in I we have

$$\begin{aligned}1 &= (\mathbf{a}^r + a_0\mathbf{b}')\cdot(\mathbf{y}' + \mathbf{t}) + s\\ &= (\mathbf{a}^r + a_0\mathbf{b}')\cdot(\mathbf{y}' + \mathbf{t}) + (\mathbf{a}^r + a_0\mathbf{b}')\cdot\mathbf{x}^r s + a_0bs\\ &= (\mathbf{a}^r + a_0\mathbf{b}')\cdot(\mathbf{y}' + \mathbf{t} + \mathbf{x}^r s) + a_0bs.\end{aligned} \tag{5}$$

Using Lemma 2.1 on (5) we find rows $\mathbf{u}$ in R^d and $\mathbf{v}$ in $(Ra_0bsR)^d \subset I^d$ such that

$$1 = (\mathbf{a}^r + a_0\mathbf{b}' + a_0bs\mathbf{u})\cdot(\mathbf{y}' + \mathbf{t} + \mathbf{v})\,.$$

Clearly the rows $\mathbf{b} = \mathbf{b}' + bs\mathbf{u}$ and $\mathbf{y} = \mathbf{y}' + \mathbf{t} + \mathbf{v}$ verify the desired conditions. □

Theorem 2.3. *If R is the surjective inverse limit of a sequence (R_n) of rings with $\mathrm{bsr}(R_n) \le d$ for all n, then also $\mathrm{bsr}(R) \le d$.*

Proof. Let $\pi_n\colon R_n \to R_{n-1}$ with $R_0 = R_1$ and $\pi_1 = \mathrm{id}$, and consider an equation $\mathbf{a}\cdot\mathbf{x} = 1$ in R^{d+1}. Identify the rows $\mathbf{a}$ and $\mathbf{x}$ with strings $(\mathbf{a}_n)$ and $(\mathbf{x}_n)$ in $\prod R_n^{d+1}$. Write $\mathbf{a}_n = (a_{n,0},\ldots,a_{n,d})$. Since $\mathrm{bsr}(R_1) \le d$ there are rows $\mathbf{b}_1$ and $\mathbf{y}_1$ in R_1^d such that $(\mathbf{a}_1^r + a_{1,0}\mathbf{b}_1)\cdot\mathbf{y}_1 = 1$. Assume now for some n that we have found rows $\mathbf{b}_k$ and $\mathbf{y}_k$ in R_k^d for $1 \le k \le n$ such that $(\mathbf{a}_k^r + a_{k,0}\mathbf{b}_k)\cdot\mathbf{y}_k = 1$, and moreover $\pi_k(\mathbf{b}_k) = \mathbf{b}_{k-1}$ and $\pi_k(\mathbf{y}_k) = \mathbf{y}_{k-1}$ for all k. By Lemma 2.2 there are rows $\mathbf{b}_{n+1}$ and $\mathbf{y}_{n+1}$ in R_{n+1}^d such that $(\mathbf{a}_{n+1}^r + a_{n+1,0}\mathbf{b}_{n+1})\cdot\mathbf{y}_{n+1} = 1$, and moreover $\pi_{n+1}(\mathbf{b}_{n+1}) = \mathbf{b}_n$ and $\pi_{n+1}(\mathbf{y}_{n+1}) = \mathbf{y}_n$. By induction we can then define rows $\mathbf{b} = (\mathbf{b}_n)$ and $\mathbf{y} = (\mathbf{y}_n)$ in R^d such that $(\mathbf{a}^r + a_0\mathbf{b})\cdot\mathbf{y} = 1$ in R, which proves that $\mathrm{bsr}(R) \le d$, as desired. □

Corollary 2.4. *Let $R = \varprojlim R_n$, where all the morphisms $R_n \to R_{n-1}$ are surjective. Then*

$$\mathrm{bsr}(R) = \sup_n \mathrm{bsr}(R_n)\,.$$

Proof. For each n the coordinate projections $\rho_n \colon R \to R_n$ are surjective, whence $\mathrm{bsr}(R_n) \leq \mathrm{bsr}(R)$ (see, e.g. [37, Theorem 4]). Thus $\sup_n \mathrm{bsr}(R_n) \leq \mathrm{bsr}(R)$.

For the converse inequality, we may of course assume that $\mathrm{bsr}(R_n) \leq d$ for all n (and some finite d). By Theorem 2.3 $\mathrm{bsr}(R) \leq d$ as well. □

3. $QB-$Rings

As shown in [6] a $QB-$ring is an infinite version of a Bass ring, i.e. a ring R with $\mathrm{bsr}(R) = 1$. To arrive at the definition we replace the set R^{-1} of invertible elements in R with the set R_q^{-1} of *quasi-invertible* elements, where $u \in R_q^{-1}$ if $(1-vu)R(1-uw) = 0 = (1-uw)R(1-vu)$ for some v, w in R. (We then write $1 - vu \perp 1 - uw$.) It follows easily that u is a (von Neumann) regular element in R, and that one may take $v = w$ and demand that u and v are partial inverses for one another (i.e. $u = uvu$ and $v = vuv$). We refer to this situation by saying that v is a *quasi-inverse* for u. The ring R is then a *quasi-Bass ring* (a $QB-$ring for short) if whenever $ax + b = 1$ in R we can find y in R such that $a + by \in R_q^{-1}$, cf. the definition of Bass rings in the introduction to Section 2. As with the notion of stable rank, the concept of $QB-$rings is left-right symmetric, see [6, Theorem 3.6].

The theory of $QB-$rings has been developed in the papers [6], [7] and [8] with the aim of extending as much as possible of the theory of Bass rings. The inspiration is the corresponding series of papers [12], [13], [14], [15] and [16], in which the theory of C^*-algebras of topological stable rank one is being extended to the class of *extremally rich* C^*-algebras, which are the C^*-analogues of $QB-$rings.

We are going to prove that any surjective inverse limit of $QB-$rings is again a $QB-$ring. To establish this in full detail turns out to be quite intricate, and we would have preferred an easier, more accessible proof. The basic techniques come from [8, Lemmas 1.4 and 1.5], and we include part of the discussion carried out there. The following Proposition examines the situation in an easier setting, hence gives an idea of what is going on.

In Theorem 3.3 below we are going to use the left-handed version of the concept of a $QB-$ring. This is done in order to make the techniques developed in [8] readily accessible. Although it will not be strictly necessary for Proposition 3.1 we have chosen the same version there to avoid confusion.

For the convenience of the reader we are only going to prove the unital versions of these results. The non-unital versions follow by straightforward, but sometimes quite exasperating computations, where invertible and quasi-invertible elements are replaced by *adversible* and *quasi-adversible* elements à la Kaplansky, cf. [6, Section 4]. Basically an element u in a non-unital ring R is (left/right/quasi) adversible if $1 - u$ is (left/right/quasi) invertible in some (hence any) unital ring $\widetilde{R}$ containing R as an ideal.

Recall that a non-unital ring I has *stable rank one* if whenever $(1 - x)(1 - a) + b = 1$ in I^+, where x, $a \in I$, we can find y in I such that $1 - a + yb \in (I^+)^{-1}$.

Proposition 3.1. *Let (R_n) be a sequence of unital $QB-$rings, and assume that we have surjective morphisms $\pi_n : R_n \to R_{n-1}$ such that* $\ker \pi_n$ *has stable rank one for every n. Then $R = \varprojlim R_n$ is also a $QB-$ring.*

Proof. Arguing along the lines of Theorem 1.4 and Theorem 2.3, it suffices to consider the case of a surjective morphism $\pi \colon R \to S$, where R and S are unital $QB-$rings and $I = \ker \pi$ has stable rank one, along with equations $xa + b = 1$ in R and $\overline{x}\,\overline{a} + \overline{b} = 1$ in S, so that $\pi(a) = \overline{a}$, $\pi(b) = \overline{b}$ and $\pi(x) = \overline{x}$. Since S is a $QB-$ring, there are elements $\overline{y}$ in S and $\overline{u}$ in S_q^{-1} such that $\overline{a} + \overline{y}\overline{b} = \overline{u}$. We wish to find elements y in R and u in R_q^{-1} such that $\pi(y) = \overline{y}$, $\pi(u) = \overline{u}$ and $a + yb = u$.

First, by [6, Proposition 7.1] lift $\overline{u}$ to a quasi-invertible element u_1 in R, and let y_1 be any lift of $\overline{y}$. We then get

$$a + y_1 b = u_1 + t \,,$$

where $t \in \ker \pi = I$. Next, choose a quasi-inverse v_1 for u_1. By computation

$$x(u_1 + t) + (1 - xy_1)b = x(u_1 + t - y_1 b) + b = xa + b = 1 \,,$$

so, multiplying left and right with u_1 and v_1 we have

$$u_1 x(u_1 v_1 + t v_1) + u_1(1 - xy_1)b v_1 = u_1 v_1 \,.$$

Rearranging terms this gives the equation

$$u_1 x(1 + t v_1) + u_1(1 - xy_1)b v_1 + (1 - u_1 x)(1 - u_1 v_1) = 1 \,.$$

By [6, Lemma 4.6] we can choose t_1 and s_1 both in I such that

$$(1 + t_1)(1 + t v_1) + s_1 \left(u_1(1 - xy_1)b v_1 + (1 - u_1 x)(1 - u_1 v_1)\right) = 1 \,.$$

Since I has stable rank one we can find s_2 in I such that

$$w = (1 + t v_1) + s_2 s_1 \left(u_1(1 - xy_1)b v_1 + (1 - u_1 x)(1 - u_1 v_1)\right) \in (I^+)^{-1} \,,$$

and we note that $1 - w \in I$. It follows that we can write

$$w u_1 = u_1 + t v_1 u_1 + s b v_1 u_1 + 0 = u_1 + (t + sb) v_1 u_1 \,,$$

where $s = s_2 s_1 u_1 (1 - xy_1) \in I$. By [6, Theorem 2.3] we have that the element

$$u_2 = u_1 + w^{-1}(t + sb)(1 - v_1 u_1) \in R_q^{-1} \,.$$

Consequently also

$$u = w u_2 = w u_1 + (t + sb)(1 - v_1 u_1) \in R_q^{-1} \,.$$

Moreover, $\pi(u) = \pi(u_2) = \pi(u_1) = \overline{u}$. Define $y = y_1 + s$. Then $= \pi(y) = \overline{y}$ and

$$\begin{aligned} a + yb &= a + y_1 b + sb = u_1 + t + sb \\ &= u_1 + (t + sb) v_1 u_1 + (t + sb)(1 - v_1 u_1) \\ &= w u_1 + (t + sb)(1 - v_1 u_1) = u \,, \end{aligned}$$

as desired. ∎

Let R be a unital ring with an ideal I. Consider an element u in R_q^{-1} with quasi-inverse v, so that $u = uvu$ and $v = vuv$, and if $p = uv$ and $q = vu$ then $(1-p) \perp (1-q)$. Let w be an element in $(qRq)_q^{-1}$ such that $q - w \in I$, and choose a quasi-inverse w', satisfying the analogous equations $w = ww'w$, $w' = w'ww'$ and $(q - ww') \perp (q - w'w)$ (whence also $q - w' \in I$). Observe that $p - uww'v = u(q - ww')v$ is an idempotent equivalent to $q - ww'$, whence $(p - uww'v) \perp (q - w'w)$. Set

$$p_1 = 1 - p, \quad p_2 = p - uww'v, \quad q_1 = q - w'w, \quad \text{and} \quad q_2 = 1 - q,$$

and note that p_2, $q_1 \in I$.

Lemma 3.2. *Let I be an ideal in a unital $QB-$ring R. If $u \in R_q^{-1}$ and $w \in (qRq)_q^{-1}$ as in the setup above, we consider an element $a = uw + t_1 + t_2$, where $t_i \in p_iRq_i$ for $i = 1, 2$. If $xa + b = 1$ for some elements x and b in R, then there is an element y in I such that $a + yb \in R_q^{-1}$.*

Proof. This follows from a verbatim repetition of the arguments in [8, Lemma 1.4 (ii)], using the additional assumption that $p_iRq_i \subset I$ for $i = 1$, 2. Observe also that by [8, Theorem 1.6] (cf. also [6, Lemma 6.1]) we can use the following fact: either $p_i \perp q_i$, or else p_iRq_i is a $QB-$corner (for each i) in the sense of [6, Definition 5.2]. □

Theorem 3.3. *If R is the surjective inverse limit of a sequence (R_n) of unital $QB-$rings, then R is also a $QB-$ring.*

Proof. As in Proposition 3.1 we only need to consider the case of a surjective morphism $\pi : R \to S$, where R and S are unital $QB-$rings, along with equations $xa + b = 1$ in R and $\overline{x}\,\overline{a} + \overline{b} = 1$ in S, so that $\pi(a) = \overline{a}$, $\pi(b) = \overline{b}$, $\pi(x) = \overline{x}$. Since S is a $QB-$ring, there are elements $\overline{y}$ in S and $\overline{u}$ in S_q^{-1} such that $\overline{a} + \overline{y}\overline{b} = \overline{u}$. We wish to find elements y in R and u in R_q^{-1} such that $\pi(y) = \overline{y}$, $\pi(u) = \overline{u}$ and $a + yb = u$.

Lift $\overline{u}$ to a quasi-invertible element u_1 in R (again by [6, Proposition 7.1]), and let y_1 be any lift of $\overline{y}$. We then get

$$a + y_1 b = u_1 + t\,,$$

where $t \in \ker \pi = I$. By computation,

$$x(u_1 + t) + (1 - xy_1)b = x(u_1 + t - y_1 b) + b = xa + b = 1\,.$$

If $b_1 = (1 - xy_1)b$ the above equation reads

$$x(u_1 + t) + b_1 = 1 \tag{6}$$

Next, choose a quasi-inverse v_1 of u_1, so that $(1 - u_1v_1) \perp (1 - v_1u_1)$ and $u_1 = u_1v_1u_1$, $v_1 = v_1u_1v_1$. Let $p = u_1v_1$ and $q = v_1u_1$. The computations carried out in [8, Lemma 1.5] can be performed in this situation also (to the equation (6)). Hence we obtain an element w in $(qRq)_q^{-1} \cap (q + qIq)$ and an element z in I such that, if w' is any quasi-inverse for w, and if we let $p_1 = 1 - p$, $p_2 = p - u_1ww'v_1$, $q_1 = q - w'w$ and $q_2 = 1 - q$, then

$$u_1 + t + u_1 zqb_1 = w_1 a_1 w_2\,,$$

where w_1, w_2 are invertible elements in R, and $a_1 = u_1 w + t_1 + t_2$, with t_i in p_iRq_i for $i = 1, 2$.

Since $x(u_1+t)+b_1=1$, we also have

$$x(u_1+t+u_1zqb_1)+(1-xu_1zq)b_1=1\,.$$

Conjugating with w_2 we get

$$(w_2xw_1)a_1+w_2(1-xu_1zq)b_1w_2^{-1}=1\,.$$

By Lemma 3.2, there is an element $y_2\in I$ such that

$$a_1+y_2w_2(1-xu_1zq)b_1w_2^{-1}\in R_q^{-1}\,,$$

whence $w_1a_1w_2+w_1y_2w_2(1-xu_1zq)b_1\in R_q^{-1}$. This means that the element

$$u=u_1+t+u_1zqb_1+w_1y_2w_2(1-xu_1zq)b_1\in R_q^{-1}\,,$$

and also $\pi(u)=\pi(u_1)=\overline{u}$. Now let $y=y_1+u_1zq(1-xy_1)+w_1y_2w_2(1-xu_1zq)(1-xy_1)$. Then $\pi(y)=\overline{y}$, and

$$\begin{aligned}a+yb&=a+y_1b+u_1zqb_1+w_1y_2w_2(1-xu_1zq)b_1\\&=u_1+t+u_1zqb_1+w_1y_2w_2(1-xu_1zq)b_1=u\,,\end{aligned}$$

as desired. □

4. Approximate Units

$\boldsymbol{\sigma}-$Unital Rings. Contrary to popular belief rings do not come automatically equipped with a unit. Of course, a unit can be adjoined, passing from R to R^+ as decribed before, but his may destroy other desirable relations for the ring, like being an ideal in a larger ring or being a Bass stable ring or a $QB-$ring. Also, indiscriminate adjoining of units may deprive the category of rings some of the applications of general category theory.

For some rings adjoining of a unit may be unneccesary because the ring itself already possesses elements that locally act as units. As in [9] a net $(e_\lambda)_{\lambda\in\Lambda}$ in R is called an *approximate unit* for R if eventually $e_\lambda x=xe_\lambda=x$ for every x in R. We do not expect that the net consists of idempotents, but in many cases it will (for example, in the case where R is an exchange ring). Far more important is the case where we can choose $\Lambda=\mathbb{N}$. We say in this case that R is $\sigma-$*unital*. Evidently we may then choose the approximate unit (e_n) such that $e_{n+1}e_n=e_ne_{n+1}=e_n$ for all n, see [9, Lemma 1.5]. Note that for such a sequence to be an approximate unit it suffices to show that for each x in R there is *some* n such that $e_nx=xe_n=x$. The same equations will then automatically hold for all indices larger than n. Thus in what follows we shall often assume that a $\sigma-$unit (e_n) satisfies $e_ne_{n+1}=e_{n+1}e_n=e_n$ for all n.

Rings with approximate units are examples of $s-$*unital* rings (see e.g. [36]). If the ring is countable it is $\sigma-$unital if and only if it is $s-$unital, see [2, Lemma 2.2].

It should also be remarked that all C^*-algebras and all classical (non-unital) Banach algebras have approximate units in the topological sense, which implies that certain dense ideals – representing elements with compact support or finite rank – have approximate units in the algebraic sense. It is instructing though to note that the ring $\mathbb{B}_f(\mathcal{H})$ of operators with finite rank on an infinite dimensional Hilbert space $\mathcal{H}$ does not have a countable approximate

unit (although it certainly has an (uncountable) approximate unit consisting of projections). By contrast, the ring $\mathbb{M}_\infty(R) = \varinjlim \mathbb{M}_n(R)$ of finite matrices over a unital or just $\sigma-$unital ring R is $\sigma-$unital.

Proposition 4.1. *Let $0 \to I \to R \to S \to 0$ be a short exact sequence of rings. If both I and S are $\sigma-$unital then also R is $\sigma-$unital.*

Proof. Let $\pi\colon R \to S$ denote the quotient map and choose an approximate unit (e_n) for I. Choose also a sequence (f_n) in R such that $(\pi(f_n))$ is an approximate unit for S.

For each x in R there is an n such that $\pi(f_n x) = \pi(x f_n) = \pi(x)$. Thus $(1 - f_n)x \in I$ and $x(1 - f_n) \in I$, so for a suitable m we have $(1 - e_m)(1 - f_n)x = 0$ and $x(1 - f_n)(1 - e_m) = 0$. It follows that if we define u_{nm} in R by

$$1 - u_{nm} = (1 - f_n)(1 - e_m)(1 - f_n)\,,$$

then $u_{nm}x = xu_{nm} = x$ for some (n, m) in $\mathbb{N}^2$. (Note here that the use of 1 is purely formal.)

We claim that for each (p, q) in $\mathbb{N}^2$ there is an $n > p$ and $m > q$ such that

$$u_{nm}u_{kl} = u_{kl}u_{nm} = u_{kl}$$

for all $k \le p$ and $l \le q$. This is equivalent to the claim that

$$\big((1 - f_n)(1 - e_m)(1 - f_n)\big)\big((1 - f_k)(1 - e_l)(1 - f_k)\big) = (1 - f_n)(1 - e_m)(1 - f_n),$$

together with the similar adjoint version. Note now that $(1 - f_k)(1 - e_l)(1 - f_k) = 1 - a_{kl}$ for some elements a_{kl} in R. Choosing n sufficiently large we may assume that $(1 - f_n)a_{kl} \in I$ for all $k \le p$ and $l \le q$. For a sufficiently large m we therefore have $(1 - e_m)(1 - f_n)a_{kl} = 0$ for all $k \le p$ and $l \le q$. Consequently,

$$\begin{aligned}&\big((1 - f_n)(1 - e_m)(1 - f_n)\big)\big((1 - f_k)(1 - e_l)(1 - f_k)\big)\\ &= (1 - f_n)(1 - e_m)(1 - f_n)(1 - a_{kl}) = (1 - f_n)(1 - e_m)(1 - f_n),\end{aligned}$$

as desired. The proof for the adjoint version is quite similar.

Using the claim we now inductively choose a sequence (v_i) in R with $v_i = u_{n(i)m(i)}$ such that $v_i u_{nm} = u_{nm} v_i = u_{nm}$ for all $n \le i$ and $m \le i$; but also such that $v_{i+1}v_i = v_i v_{i+1} = v_i$ for all i. As we saw above, there is for each x in R some (n, m) in $\mathbb{N}^2$ such that $u_{nm}x = xu_{nm} = x$, and it follows that if $i \ge \max\{n, m\}$ then

$$v_i x = v_i u_{nm} x = u_{nm} x = x,$$

and similarly $xv_i = x$. Thus (v_i) is a $\sigma-$unit for R. □

Definition 4.2. If I is an ideal in a ring R we say that I *has a unit in* R if $ex = xe = x$ for some e in R and every x in I. Similarly we say that I *has a $\sigma-$unit in* R if there is a sequence e_n in R such that eventually $e_n x = xe_n = x$ for each x in I.

Inspection of the proof of Proposition 4.1 shows that in order for R to be $\sigma-$unital it suffices that S is $\sigma-$unital and I has a $\sigma-$unit in R. In this formulation the requirements are both necessary and sufficient.

The argument also shows that R is unital if and only if S is unital and I has a unit in R. If e is a unit for I in R and f is an element in R such that $\pi(f)$ is the unit in S then the element u in R given by $1 - u = (1 - f)(1 - e)(1 - f)$ is the unit in R.

Theorem 4.3. *let $R = \varprojlim R_n$ be a surjective inverse limit of a sequence of rings with coordinate evaluations $\rho_n \colon R \to R_n$. Then R is $\sigma-$unital if and only if each R_n is $\sigma-$unital and the ideals* $\ker \rho_n$ *have a unit in R eventually.*

Proof. If R is $\sigma-$unital then so is every R_n, since the morphisms ρ_n are surjective. The ideals $\ker \rho_n$ form a decreasing sequence, so if they do not eventually have a unit in R none of them will. To obtain a contradiction, let us assume this to be the case, and let (e_n) be an approximate unit for R.

Given x_1 in R_1 we claim that there is an x in R such that $\rho_1(x) = x_1$ and either $e_2 x \neq x$ or $xe_2 \neq x$. For if this were not the case then

$$e_2(x+s) = (x+s)e_2 = x+s$$

for each fixed x in R with $\rho_1(x) = x_1$ and every s in $\ker \rho_1$. In particular, e_2 would be a unit for $\ker \rho_1$ in R, contradicting our assumptions. Assuming therefore, as we may, that $e_2 x \neq x$ we can find a (first) coordinate k_2 in which they differ. Thus, if we let $x_2 = \rho_{k_2}(x)$, then $\rho_{k_2}(e_2)x_2 \neq x_2$ and $\pi_2 \circ \cdots \circ \pi_{k_2}(x_2) = x_1$. By the same argument we can now find an element y in R with $\rho_{k_2}(y) = x_2$, such that either $e_3 y \neq y$ or $ye_3 \neq y$. This produces a coordinate number k_3 and an element $x_3 = \rho_{k_3}(x)$ such that either $\rho_{k_3}(e_3)x_3 \neq x_3$ or $x_3\rho_{k_3}(e_3) \neq x_3$ and $\pi_{k_2-1} \circ \cdots \pi_{k_3}(x_3) = x_2$. Continuing by induction we find a coherent sequence $x = (x_n)$ in (R_{k_n}), i.e. $\pi_{k_n-1} \circ \cdots \circ \pi_{k_{n+1}}(x_{n+1}) = x_n$ for all n, such that for each n either $\rho_{k_n}(e_n)x_n \neq x_n$ or $x_n \rho_{k_n}(e_n) \neq x_n$. Identifying $x = (x_n)$ with an element in R we see that either $e_n x \neq x$ or $xe_n \neq x$ for every n, contradicting the assumption that (e_n) was an approximate unit.

In the converse direction, if R_k is $\sigma-$unital and $\ker \rho_k$ has a unit in R, then R is $\sigma-$unital by Proposition 4.1 and the comments after Definition 4.2 applied to the extension

$$0 \to \ker \rho_k \to R \to R_k \to 0\,.$$

□

The Theorem above shows that in all but a few cases a surjective inverse limit of non-unital rings will *not* be $\sigma-$unital. The demand of an eventual unit for $\ker \rho_n$ in R translates to the demand that with the morphisms $\pi_m \colon R_m \to R_{m-1}$ there is a unit e_m for $\ker \sigma_m$ in R_m for each $m > n$, where $\sigma_m = \pi_{n+1} \circ \pi_{n+2} \circ \cdots \circ \pi_m$, and moreover these can be chosen coherent, i.e. $\pi_{m+1}(e_{m+1}) = e_m$ for all m. Clearly this is not likely to happen in actual examples.

5. Multiplier Rings

Centralizers. Throughout this section we assume that R is a *non-degenerate* ring, i.e. $xR = 0$ or $Rx = 0$ implies $x = 0$ for every x in R. This concept is of course only relevant for non-unital rings, but note that rings with approximate units will automatically be non-degenerate. Eventually we are going to need the stronger concept that R is *semi-prime*, which means that $xRx = 0$ implies $x = 0$ for every x in R.

Following Hochschild, [22, Definition 3.1], although using the terminology from [23], we define a *left centralizer* to be a map $\lambda \colon R \to R$ such that $\lambda(xy) = \lambda(x)y$ for all x, y in R. Similarly, a *right centralizer* ρ is a map such that $\rho(xy) = x\rho(y)$. A *double centralizer* is a pair (λ, ρ) of left-right centralizers satisfying the coherence relation $x\lambda(y) = \rho(x)y$ for all

x, y in R. It is easy to see that (in the presence of non-degeneracy) this relation alone will imply that λ and ρ are both $R-$module maps, cf. [23, Theorem 7]. For such pairs we define a product by setting

$$(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_1 \circ \lambda_2, \rho_2 \circ \rho_1)\,.$$

With the obvious sum $(\lambda_1, \rho_1)+(\lambda_2, \rho_2) = (\lambda_1+\lambda_2, \rho_1+\rho_2)$ the set $\mathcal{M}(R)$ of double centralizers becomes a unital ring with the unit $1 = (\mathrm{id}, \mathrm{id})$. We shall refer to $\mathcal{M}(R)$ as the *multiplier ring* for R.

For each x in R we define the left and right centralizers λ_x and ρ_x by $\lambda_x(y) = xy$ and $\rho_x(y) = yx$. Then (λ_x, ρ_x) is a double centralizer, and since $(\lambda_x, \rho_x)(\lambda_y, \rho_y) = (\lambda_{xy}, \rho_{xy})$ it follows that we have a morphism $x \mapsto (\lambda_x, \rho_x)$ of R into $\mathcal{M}(R)$. The non-degeneracy of R ensures that this morphism is injective, so we may identify R with its image in $\mathcal{M}(R)$. The easily verified formulas

$$\lambda \circ \lambda_x = \lambda_{\lambda(x)}, \quad \lambda_x \circ \lambda = \lambda_{\rho(x)}, \quad \rho \circ \rho_x = \rho_{\rho(x)}, \quad \rho_x \circ \rho = \rho_{\lambda(x)},$$

show that

$$(\lambda, \rho)(\lambda_x, \rho_x) = (\lambda_{\lambda(x)}, \rho_{\lambda(x)}), \quad (\lambda_x, \rho_x)(\lambda, \rho) = (\lambda_{\rho(x)}, \rho_{\rho(x)})\,, \tag{7}$$

which proves that R is a two-sided ideal in $\mathcal{M}(R)$. At the same time we see from (7) that R is an *essential ideal* in $\mathcal{M}(R)$, because $R(\lambda, \rho) = 0$ or $(\lambda, \rho)R = 0$ for some (λ, ρ) in $\mathcal{M}(R)$ implies that $\lambda = 0$ or $\rho = 0$, which forces $(\lambda, \rho) = 0$ by the coherence relation. It is easy to show that the multiplier ring has the universal property that if S is any ring containing R as an ideal there is a morphism $\varphi\colon S \to \mathcal{M}(R)$ extending the embedding morphism $R \to S$. Moreover, φ is injective precisely when R is an essential ideal in S. For each y in S one just defines $\varphi(y) = (\lambda_y, \rho_y)$.

The multiplier ring $\mathcal{M}(R)$, or rather its quotient $\mathcal{M}(R)/R$, is an indispensable tool in Hochschild's classification of extensions of R, see the remarks at the end of the paper. It also occurs naturally as the function-theoretic method to describe the Stone-Čech compactification of a (locally compact) topological space. In $K-$theory it is used to describe the ring $\mathbb{B}(R)$ of row- and column-finite matrices over a unital ring R. Specifically, if $\mathbb{M}_\infty(R) = \varinjlim \mathbb{M}_n(R)$, identified with the ring of finite matrices over R, we find that its multiplier ring is identified with $\mathbb{B}(R)$ (see e.g. [9, Proposition 1.1]).

Definition 5.1. A morphism $\pi\colon R \to S$ between rings R and S is called *proper* of $\pi(R)$ is not contained in any proper left or right ideal of S, i.e. $S\pi(R) = \pi(R)S = S$ (in the sense that $\pi(R)S$ denotes the set of finite sums of products $\pi(x)y$). If R is unital then π is proper if and only if both S and π are also unital.

The origin of this notion comes from C^*-algebra theory, cf. [17] or [26], where one notices that if X and Y are locally compact Hausdorff spaces and $R = C_0(X)$ and $S = C_0(Y)$ denote the rings of complex-valued continuous functions vanishing at infinity, then a morphism (which in this category means a *-homomorphism) $\pi\colon R \to S$ is proper if and only if it is the transposed of a *proper continuous map* $f\colon Y \to X$, i.e. one for which $f^{-1}(C)$ is compact in Y for every compact subset C in X.

Theorem 5.2. *Let $\pi\colon R \to S$ be a proper morphism between non-degenerate rings R and S. There is then a unique unital morphism $\overline{\pi}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ that extends π.*

Proof. If $(\lambda, \rho) \in \mathcal{M}(R)$ and y, z are elements in S we can write them in the form $y = \sum \pi(u_i)y_i$ and $z = \sum z_j\pi(v_j)$ for suitable finite subsets $\{u_i\}, \{v_j\}$ in R and $\{y_i\}, \{z_j\}$ in S. Now define

$$\overline{\lambda}(y) = \sum \pi(\lambda(u_i))y_i \quad \text{and} \quad \overline{\rho}(z) = \sum z_j\pi(\rho(v_j)) .$$

If $x \in R$ then

$$\pi(x)\overline{\lambda}(y) = \sum \pi(x\lambda(u_i)y_i) = \sum \pi(\rho(x)u_i)y_i = \pi(\rho(x)) \sum \pi(u_i)y_i = \pi(\rho(x))y .$$

Similarly, $\overline{\rho}(z)\pi(x) = z\pi(\lambda(x))$. For one thing, this implies that the elements $\overline{\lambda}(y)$ and $\overline{\rho}(z)$ are independent of the representations of y and z. If namely $\overline{\lambda'}(y)$ arises from another representation $y = \sum \pi(u_i')y_i'$, then $\pi(x)(\overline{\lambda}(y) - \overline{\lambda'}(y)) = \pi(\rho(x))(y - y) = 0$ for all x in R, which forces $\overline{\lambda}(y) = \overline{\lambda'}(y)$ by properness and non-degeneracy. On the other hand the equations show that we have the coherence relation

$$z\overline{\lambda}(y) = \sum z_j\pi(v_j)\overline{\lambda}(y) = \sum z_j\pi(\rho(v_j))y = \overline{\rho}(z)y .$$

We may therefore define $\overline{\pi}(\lambda, \rho) = (\overline{\lambda}, \overline{\rho})$ in $\mathcal{M}(S)$, and it is straightforward to check that $\overline{\pi}$ is a unital morphism from $\mathcal{M}(R)$ to $\mathcal{M}(S)$. By construction $\overline{\pi}$ is an extension of π, and since S is an essential ideal of $\mathcal{M}(S)$ there can be only one extension, so $\overline{\pi}$ is unique. □

The Strict Topology. If R is a non-degenerate ring with multiplier ring $\mathcal{M}(R)$ we can for each a in R define

$$\mathcal{O}_a = \{x \in M(R) \mid xa = ax = 0\}.$$

As shown in [9] the sets $\mathcal{O}_a$ form a subbasis for the neighbourhood system around 0 in the *strict topology* on $\mathcal{M}(R)$. In this topology addition is continuous (by construction) and multiplication is continuous by [9, Lemma 1.3]. A *strict Cauchy* net (x_λ) in $\mathcal{M}(R)$ is a net such that each product net (ax_λ) or $(x_\lambda a)$ is eventually constant. It follows easily from this that $\mathcal{M}(R)$ is strictly complete, cf. [9, Proposition 1.6].

If R has an approximate unit (e_λ), then $e_\lambda \to 1$ strictly in $\mathcal{M}(R)$, and R is therefore strictly dense in $\mathcal{M}(R)$. Conversely, if 1 is a strict limit point for R then every net (e_λ) in R converging strictly to 1 will be an approximate unit for R, and R is strictly dense in $\mathcal{M}(R)$, cf. [9, Proposition 1.6].

Proposition 5.3. *If $\pi\colon R \to S$ is a proper morphism between non-degenerate rings R and S then the extension $\overline{\pi}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ from Theorem 5.2 is strictly continuous.*

Proof. Let (x_λ) be a net in $\mathcal{M}(R)$ converging strictly to 0. Given y in S we can then by properness write $y = \sum \pi(x_i)y_i$. Consequently

$$\overline{\pi}(x_\lambda)y = \sum \pi(x_\lambda x_i)y_i = 0$$

eventually. Similarly $y\overline{\pi}(x_\lambda) = 0$ eventually, which proves that $\overline{\pi}(x_\lambda) \to 0$ strictly in $\mathcal{M}(S)$. □

The next theorem is the algebraic counterpart of a rather useful result from C^*-algebra theory, known as the *non-commutative Tietze extension theorem*, cf. [31, Proposition 3.12.10] or [32, Theorem 10]. Applied to the ring $R = C_0(X)$ of complex-valued continuous functions

on some locally compact Hausdorff space X, we find that $\mathcal{M}(R) = C_b(X)$, the ring of bounded continuous functions on X. The content of the theorem is then that for any bounded continuous function f on a closed subset Y of X (and these are the only ways quotients of the ring $C_0(X)$ can arise), there is an extension $\overline{f}$ of f to a bounded continuous function on all of X. The necessity of a *countable* approximate unit for R reflects the fact that a general locally compact space X need not be normal (and only for normal spaces will the Tietze extension theorem hold), but if in addition X is $\sigma-$compact, *then* it is normal.

Theorem 5.4. *Let $\pi\colon R \to S$ be a surjective morphism between $\sigma-$unital rings. Then the extension $\overline{\pi}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ from Theorem 5.2 is also surjective.*

Proof. Let (e_n) be a countable approximate unit for R, and without loss of generality assume that $e_{n+1}e_n = e_n = e_n e_{n+1}$ for all n. Consider now an element $\overline{x}$ in $\mathcal{M}(S)$. Since the elements $\overline{e}_n = \pi(e_n)$ form an approximate unit for S we have for each n that $\overline{e}_m \overline{x}\, \overline{e}_n = \overline{x}\, \overline{e}_n$ for m large enough. Passing to a subsequence we may therefore assume that

$$\overline{e}_{n+1}\overline{x}\,\overline{e}_n = \overline{x}\,\overline{e}_n \tag{8}$$

for all n.

Assume that for some $n \geq 2$ we have found elements $x_1, \ldots, x_n$ in R with $\pi(x_k) = \overline{e}_k\overline{x}$ such that with $e_{-1} = e_0 = 0$ we have

$$(x_k - x_{k-1})e_{k-3} = 0 = e_{k-3}(x_k - x_{k-1}) \tag{9}$$

for $2 \leq k \leq n$. Choose z in R with $\pi(z) = \overline{e}_{n+1}\overline{x}$. Then by (8)

$$\pi((z - x_n)e_{n-1}) = (\overline{e}_{n+1} - \overline{e}_n)\overline{x}\,\overline{e}_{n-1} = 0\,,$$
$$\pi(e_{n-1}(z - x_n)) = \overline{e}_{n-1}(\overline{e}_{n+1} - \overline{e}_n)\overline{x} = 0\,.$$

Thus $(z - x_n)e_{n-1} \in \ker \pi$ and $e_{n-1}(z - x_n) \in \ker \pi$. Now define

$$x_{n+1} = z - (z - x_n)e_{n-1} - e_{n-1}(z - x_n)(1 - e_{n-1})\,,$$

so that $\pi(x_{n+1}) = \overline{e}_{n+1}\overline{x}$. Moreover,

$$\begin{aligned} x_{n+1} - x_n &= (z - x_n)(1 - e_{n-1}) - e_{n-1}(z - x_n)(1 - e_{n-1}) \\ &= (1 - e_{n-1})(z - x_n)(1 - e_{n-1})\,. \end{aligned}$$

This means that

$$e_{n-2}(x_{n+1} - x_n) = 0 = (x_{n+1} - x_n)e_{n-2}\,.$$

By induction we can therefore find a sequence (x_n) in R satisfying (9) such that also $\pi(x_n) = \overline{e}_n\overline{x}$ for all $n \geq 3$.

For each a in R there is a number k such that $e_k a = a = a e_k$. It follows from (9) that

$$(x_m - x_n)a = (x_m - x_n)e_k a = 0\,, \quad \text{and similarly} \quad a(x_m - x_n) = 0$$

for all $n, m \geq k+2$. Consequently (x_n) is a strict Cauchy sequence in $\mathcal{M}(R)$, hence convergent to an element x in $\mathcal{M}(R)$. Since $\overline{\pi}$ is strictly continuous by Proposition 5.3 we finally see that

$$\overline{\pi}(x) = \lim \pi(x_n) = \lim \overline{e}_n\overline{x} = \overline{x}\,,$$

as desired. ∎

6. Multiplier Rings as Inverse Limits

As before we let $R = \varprojlim R_n$ for a sequence of rings R_n with surjective morphisms $\pi_n \colon R_n \to R_{n-1}$ and consider the surjective coordinate evaluation morphisms $\rho_n \colon R \to R_n$. If I is an ideal in R we evidently obtain a sequence (I_n) of ideals, where $I_n = \rho_n(R) \subset R_n$, and by restriction this defines surjective morphisms $\pi_n \colon I_n \to I_{n-1}$. Conversely, if we have a sequence of ideals $I_n \subset R_n$ such that $\pi_n(I_n) = I_{n-1}$ for all n, we may identify the inverse limit $\varprojlim I_n$ with an ideal $\overline{I}$ in R. Note that if we start with an ideal I in R then $I \subset \overline{I}$, and in general the inclusion is strict.

Theorem 6.1. *If $\varprojlim R_n$ is the surjective inverse limit of a sequence of $\sigma-$unital rings and I is an ideal of $\varprojlim R_n$ such that $\rho_n(I) = R_n$ for every n then $\mathcal{M}(I) = \varprojlim \mathcal{M}(R_n)$.*

Proof. Put $R = \varprojlim R_n$ and $\mathcal{M} = \varprojlim \mathcal{M}(R_n)$. We then claim that there is a commutative diagram

$$\begin{tikzcd}
I \arrow[r, "\iota_0"] \arrow[d, "\iota"] & R \arrow[r, "\rho_n"] \arrow[d, "\rho"] & R_n \arrow[r, "\pi_n"] \arrow[d, "\iota_n"] & R_{n-1} \arrow[d, "\iota_{n-1}"] \\
\mathcal{M}(I) & \mathcal{M} \arrow[r, "\overline{\rho}_n"] & \mathcal{M}(R_n) \arrow[r, "\overline{\pi}_n"] & \mathcal{M}(R_{n-1})
\end{tikzcd}$$

Here ι and ι_k for $k \geq 0$ are the natural embeddings, and $\overline{\pi}_n$ is the surjective morphism obtained from Theorem 5.4. It follows that $\mathcal{M}$ is the surjective inverse limit of the multiplier rings $\mathcal{M}(R_n)$, and the coordinate evaluations $\overline{\rho}_n$ are therefore also surjective.

Since the right-hand square of the diagram is commutative we can define the morphism ρ by $(x_n) \mapsto (\iota_n(x_n))$ for every string $x = (x_n)$ in R, and we note that $\overline{\rho}_n \circ \rho = \iota_n \circ \rho_n$ by this definition. Evidently ρ is injective. If moreover $y = (y_n)$ is a string in $\mathcal{M}$ then for any x in R we have that $\overline{\rho}_n(\rho(x)y) = \iota_n(x_n)y_n = x_n y_n$. Identifying $(x_n y_n)$ with a string in R (using the commutativity of the diagram) we see that $\rho(R)$ is an ideal in $\mathcal{M}$, which must even be essential, since $\rho(R)y = 0$ implies $R_n \overline{\rho}_n(y) = \rho_n(R)\overline{\rho}_n(y) = \overline{\rho}_n(\rho(R)y) = 0$, hence $\overline{\rho}_n(y) = 0$ for all n by non-degeneracy, and so $y = 0$.

We claim that I is essential in R. For if $xI = 0$ for some x in R, then $\rho_n(x)R_n = 0$ for every n by our assumption on I, whence $\rho_n(x) = 0$ since R_n is non-degenerate (being $\sigma-$unital), and therefore $x = 0$. Since the property of being essential is hereditary it follows that $\rho(\iota_0(I))$ is an essential ideal in $\mathcal{M}$. By the universal property of multiplier rings there is therefore an injective morphism $\varphi \colon \mathcal{M} \to \mathcal{M}(I)$ such that $\varphi \circ \rho \circ \iota_0 = \iota$.

Each surjective morphism $\rho_n \circ \iota_0$ extends uniquely to a (not necessarily surjective) morphism $\psi_n \colon \mathcal{M}(I) \to \mathcal{M}(R_n)$ by Theorem 5.2. Since $\pi_n \circ \rho_n \circ \iota_0 = \rho_{n-1} \circ \iota_0$ and I is essential in $\mathcal{M}(I)$ it follows that also $\overline{\pi}_n \circ \psi_n = \psi_{n-1}$ for all n. By the universal property of inverse limits this means that we have a unique morphism $\psi \colon \mathcal{M}(I) \to \mathcal{M}$ such that $\overline{\rho}_n \circ \psi = \psi_n$ for all n. It follows that

$$\overline{\rho}_n \circ \rho \circ \iota_0 = \iota_n \circ \rho_n \circ \iota_0 = \psi_n \circ \iota = \overline{\rho}_n \circ \psi \circ \iota$$

for all n, which implies that $\psi \circ \iota = \rho \circ \iota_0$.

Combining these results we find that

$$(\varphi \circ \psi) \circ \iota = \varphi \circ \rho \circ \iota_0 = \iota \quad \text{and} \quad (\psi \circ \varphi) \circ \rho \circ \iota_0 = \psi \circ \iota = \rho \circ \iota_0\,.$$

Since $\iota(I)$ is an essential ideal in $\mathcal{M}(I)$ and $\rho(\iota_0(I))$ is an essential ideal in $\mathcal{M}$ these equations imply that φ and ψ are the inverse of one another, and we have our natural isomorphism. $\square$

Corollary 6.2. *If (R_n) is a sequence of $\sigma-$unital rings with surjective morphisms $\pi_n\colon R_n \to R_{n-1}$, and if $\overline{\pi}_n\colon \mathcal{M}(R_n) \to \mathcal{M}(R_{n-1})$ denote the unique surjective extensions of the π_n's then*

$$\mathcal{M}(\varprojlim R_n) = \varprojlim \mathcal{M}(R_n)\,.$$

O'Meara proved in [29] that if R is a $\sigma-$unital von Neumann regular ring then $\mathcal{M}(R)$ is an exchange ring (but is generally not regular). His arguments used critically the $\sigma-$unital assumption on the ring R. In the next result we show that in certain cases, multiplier rings of not necessarily $\sigma-$unital regular rings are also exchange rings. These are obtained as inverse limits of $\sigma-$unital regular rings, which will generally not be $\sigma-$unital themselves, cf. Theorem 4.3.

Corollary 6.3. *If $R = \varprojlim R_n$ is a surjective limit of a sequence of $\sigma-$unital von Neumann regular rings then $\mathcal{M}(R)$ is an exchange ring.*

Proof. If R_n is von Neumann regular and $\sigma-$unital for all n, then $\mathcal{M}(R_n)$ is an exchange ring, by [29, Theorem 2]. The conclusion now follows from Corollary 6.2 and Theorem 1.4. $\square$

Let as before $R = \varprojlim R_n$ be a surjective inverse limit of a sequence of $\sigma-$unital rings. Assume moreover that all R_n are semi-prime. This implies that also R is semi-prime. For if $xRx = 0$ for some x in R then $\rho_n(x)R_n\rho_n(x) = 0$ for every n, whence $\rho_n(x) = 0$ so that $x = 0$. If now I is an ideal in R we define the (two-sided) *annihilator ideal* by

$$I^\perp = \{x \in R \mid xI = 0\} = \{x \in R \mid Ix = 0\}\,.$$

Then $I^\perp \cap I = 0$; in fact $I^\perp$ is the largest ideal orthogonal to I. In particular, $I + I^\perp$ is essential in R, cf. [24, Exer. 6.8] or [5, 1.1.1].

Definition 6.4. Let $\varprojlim R_n$ be a surjective inverse limit of a sequence of semi-prime rings. We say that an ideal I in $\varprojlim R_n$ is *$m-$constant* if $I \cap \ker \rho_m = 0$. Equivalently, $I \subset (\ker \rho_m)^\perp$. We may graphically view $\ker \rho_m$ as the strings $x = (x_n)$ in R such that $x_n = 0$ for $n \le m$. Evidently these coordinate ideals decrease, so that if I is *$m-$constant*, then $I \cap \ker \rho_n = 0$ for all $n \ge m$.

Since $\pi_n \circ \rho_n = \rho_{n-1}$ we see that $\ker \pi_n \subset \rho_n(\ker \rho_m)$ for $n > m$. If therefore $I_n = \rho_n(I)$ denotes the associated sequence of ideals in R_n, then $I_n \cap \ker \pi_n = 0$ for $n > m$. Thus I is isomorphic to I_m and I_n is isomorphic to I_m for all $n \ge m$. In particular, $\varprojlim I_n = I$.

Conversely, if (I_n) is a sequence of ideals in (R_n) such that $\pi_n(I_n) = I_{n-1}$ for all n and $I_n \cap \ker \pi_n = 0$ for all $n \ge m$ for some m then $I = \varprojlim I_n$ will be an $m-$constant ideal in $\varprojlim R_n$.

If I is an $n-$constant and J is an $m-$constant ideal with $n \le m$, then $I+J \subset (\ker \rho_m)^\perp$ since $\ker \rho_m \subset \ker \rho_n$, so $I + J$ is an $m-$constant ideal. Thus $(\ker \rho_m)^\perp$ is the largest $m-$constant ideal. It follows that $I_c = \bigcup(\ker \rho_m)^\perp$ is equal to the sum of all constant ideals, and we shall refer to it as the *quasi-constant ideal* of R.

The motivating example for considering constant and quasi-constant ideals arises from the Stone-Čech compactification. If X is a locally compact Hausdorff space then its Stone-Čech compactification βX has the property that $C(\beta X) = C_b(X) = \mathcal{M}(C_0(X))$, and this ring is always a C^*-inverse limit. In the case where X is also $\sigma-$compact we can write $X = \bigcup X_n$, where each X_n is a compact subset of X contained in the interior X_{n+1}° of X_{n+1}. Put $R_n = C(X_n)$ and let $\pi_n(f) = f|X_{n-1}$ for each f in $C(X_n)$. Then $C_b(X) = \varprojlim R_n$. The large constant ideals will be of the form $(\ker \pi_m)^\perp = C_0(X_m^\circ)$, so the quasi-constant ideal of $C_b(X)$ can (in this category) be identified with the norm-closed ideal $C_0(X)$.

If the construction is carried out in a purely algebraic context we find that the ring $C(X)$ of all continuous functions on X is the (algebraic) inverse limit of the sequence $(C(X_n))$, and the quasi-constant ideal of $C(X)$ is the ring $C_c(X)$ of continuous functions with compact supports.

Theorem 6.1 provides an immediate generalization of this construction:

Corollary 6.5. *If $R = \varprojlim R_n$ is the surjective inverse limit of a sequence of $\sigma-$unital rings such that the quasi-constant ideal I_c of R satisfies $\rho_n(I_c) = R_n$ for every n, then $\mathcal{M}(I_c) = \varprojlim \mathcal{M}(R_n)$.*

Theorem 6.6. *Let (I_n) and (J_n) be two sequences of ideals in a semi-prime $\sigma-$unital ring R, one increasing, the other decreasing, but such that $I_n \cap J_n = 0$ for all n. If $I = \bigcup I_n$ is essential in R and $I + J_n = R$ for every n, then with $R_n = R/J_n$ and $\pi_n\colon R_n \to R_{n-1}$ the natural morphisms we have an embedding $\overline{\varphi}\colon R \to \varprojlim R_n$ such that $\overline{\varphi}(I)$ is an ideal. Moreover, $\mathcal{M}(I) = \varprojlim \mathcal{M}(R_n)$.*

Proof. If $x \in \bigcap J_n$ then it annihilates I_n for every n, whence $x \in I^\perp$. But then $x = 0$ since I is essential and R is semi-prime. Thus our assumptions imply that $\bigcap J_n = 0$.

The quotient morphisms $\varphi_n\colon R \to R_n$ satisfy $\pi_n \circ \varphi_n = \varphi_{n-1}$ for all n, and therefore define a morphism $\overline{\varphi}\colon R \to \varprojlim R_n$ such that $\rho_n \circ \overline{\varphi} = \varphi_n$ for all n, where $\rho_n : \varprojlim R_n \to R_n$ are the coordinate evaluations. Since $\ker \overline{\varphi} = \bigcap J_n = 0$, this is an embedding.

Observe that $\varphi_n(I_m) = (I_m + J_n)/J_n$ is an ideal in R_n for every n and m. Since $\ker \pi_n = J_{n-1}/J_n$ we see moreover that $\varphi_n(I_m) \cap \ker \pi_n = 0$ for $n > m$. Consequently $(\varphi_n(I_m))$ is a coherent sequence of ideals in (R_n), all isomorphic for $n \geq m$, thus giving rise to the $m-$constant ideal $\overline{\varphi}(I_m)$ in $\varprojlim R_n$.

It follows from this that $\overline{\varphi}(I)$ is an ideal in $\varprojlim R_n$, isomorphic to I (and contained in the quasi-constant ideal of $\varprojlim R_n$). Since by assumption

$$\rho_n(\overline{\varphi}(I)) = \varphi_n(I) = (I + J_n)/J_n = R_n\,,$$

it follows from Theorem 6.1 that $\mathcal{M}(I) = \varprojlim \mathcal{M}(R_n)$. □

Corollary 6.7. *Let (I_n) and (J_n) be two sequences of ideals in a semi-prime $\sigma-$unital ring R, one increasing, the other decreasing, but such that $I_n \cap J_n = 0$ for every n. If $\bigcup I_n = R$ and each quotient $R_n = R/J_n$ is unital, then with $\pi_n\colon R_n \to R_{n-1}$ the natural morphisms we have an embedding of R as an ideal in $\varprojlim R_n$ such that $\mathcal{M}(R) = \varprojlim R_n$.*

Examples 6.8. Let (e_n) be a countable approximate unit consisting of central elements in a semi-prime ring R. Assuming, as we may, that $e_n e_{n+1} = e_n$ for all n we consider the principal ideals $I_n = e_n R$ and $J_n = (1 - e_{n+1})R$ in R. Note that $e_n \in I_n$, whereas J_n consists of elements of the form $x - e_{n+1}x$, where $x \in R$. Evidently $I_n \cap J_n = 0$ and $\bigcup I_n = R$. Moreover $R_n = R/J_n$ is unital with unit $e_{n+1} + J_n$. It follows from Corollary 6.7 that $\mathcal{M}(R) = \varprojlim R_n$.

Along the same lines we may assume that the semi-prime ring R has a countable approximate unit (e_n), such that $e_{n+1}x = x = xe_{n+1}$ for every x in the principal ideal I_n generated by e_n. This could be accomplished by placing suitable finiteness conditions on I_n, for example finite dimensionality. As before we let J_n be the principal ideal in R generated by $1 - e_{n+1}$ (consisting of sums of elements of the form $x - xe_{n+1}$, $y - e_{n+1}y$ and $xy - xe_{n+1}y$, with x, y in R). By assumption $I_n \cap J_n = 0$ and $\bigcup I_n = R$. Moreover, each quotient $R_n = R/J_n$ is unital with unit $e_{n+1} + J_n$, so again $\mathcal{M}(R) = \varprojlim R_n$.

Note that in both examples the quotients $R_{n-1} = R/J_{n-1}$ can be identified with the local rings R_{e_n} ($= e_n R e_n$ as an additive group, but with the product $(e_n x e_n)(e_n y e_n) = e_n x e_n y e_n$).

Pullbacks. Since the functor $\varprojlim$ has a left adjoint it commutes with extensions and pullbacks. Thus if for each n we have a commutative diagram of rings with morphisms

$$\begin{tikzcd} R_n \arrow[r, "\alpha_n"] \arrow[d, "\pi_n"] & T_n \arrow[d, "\tau_n"] & S_n \arrow[l, "\beta_n"'] \arrow[d, "\rho_n"] \\ R_{n-1} \arrow[r, "\alpha_{n-1}"] & T_{n-1} & S_{n-1} \arrow[l, "\beta_{n-1}"'] \end{tikzcd}$$

we can form the pullback rings and obtain induced morphisms

$$R_n \oplus_{T_n} S_n \xrightarrow{\sigma_n} R_{n-1} \oplus_{T_{n-1}} S_{n-1}\,.$$

If for simplicity we put

$$R = \varprojlim R_n, \quad S = \varprojlim S_n \quad \text{and} \quad T = \varprojlim T_n$$

we also have induced morphisms $R \to T \leftarrow S$, and as in [33, Proposition 4.16] we obtain a natural isomorphism

$$\varprojlim (R_n \oplus_{T_n} S_n) = R \oplus_T S$$

Extensions. If we have a commutative diagram of extensions

$$\begin{tikzcd} I_n \arrow[r, "\iota_n"] \arrow[d, "\pi_n"] & R_n \arrow[r, "\alpha_n"] \arrow[d, "\omega_n"] & S_n \arrow[d, "\theta_n"] \\ I_{n-1} \arrow[r, "\iota_{n-1}"] & R_{n-1} \arrow[r, "\alpha_{n-1}"] & S_{n-1} \end{tikzcd} \tag{10}$$

then, again by category theory, we obtain an extension

$$\varprojlim I_n \xrightarrow{\bar{\iota}} \varprojlim R_n \xrightarrow{\bar{\alpha}} \varprojlim S_n\,. \tag{11}$$

We know from [22] that every extension is determined (up to an obvious isomorphism) by the morphism σ (the *Hochschild invariant*) occurring in the natural diagram below:

$$\begin{array}{ccccc} I & \xrightarrow{\iota} & R & \xrightarrow{\alpha} & S \\ \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\sigma} \\ I & \xrightarrow{\iota} & \mathcal{M}(I) & \xrightarrow{\beta} & \mathcal{Q}(I) \end{array}$$

Here $\mathcal{M}(I)$ is the multiplier ring of the ideal I, assumed to be non-degenerate, and $\mathcal{Q}(I)$, the corona ring, is the quotient $\mathcal{M}(I)/I$. The morphism φ is the one obtained by the universal property of $\mathcal{M}(I)$, and σ is the morphism induced by φ to make the whole diagram commutative. The salient fact to know is that the right-hand square in the diagram is a pullback.

As a consequence of our results about multipliers of inverse limits, in particular Corollary 6.2, we can compute the Hochschild invariant of an inverse limit of extensions:

Proposition 6.9. *Given a coherent sequence of extensions as in the diagram (10), each determined by a Hochschild invariant $\sigma_n\colon S_n \to \mathcal{Q}(I_n)$, where (I_n) is sequence of $\sigma-$unital ideals and all the morphisms π_n and ω_n (hence also θ_n) are surjective, then since $\mathcal{Q}(\varprojlim I_n) = \varprojlim \mathcal{Q}(I_n)$, the Hochschild invariant for the extension (11) is the induced morphism $\overline{\sigma}$ occurring in the diagram:*

$$\begin{array}{ccccc} \varprojlim I_n & \xrightarrow{\overline{\iota}} & \varprojlim R_n & \xrightarrow{\overline{\varphi}} & \varprojlim S_n \\ \| & & \downarrow{\scriptstyle\overline{\varphi}} & & \downarrow{\scriptstyle\overline{\sigma}} \\ \varprojlim I_n & \xrightarrow{\overline{\iota}} & \varprojlim \mathcal{M}(I_n) & \xrightarrow{\overline{\psi}} & \varprojlim \mathcal{Q}(I_n) \end{array}$$

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